## THE UNIVERSITY OF AKRON

## Mathematics and Computer Science

Lesson 3: Basic Algebra, Part I

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$$
\operatorname{Iam} \mathfrak{P S}
$$

$$
\begin{gathered}
\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C} \\
a^{3} a^{4}=a^{7}(a b)^{10}=a^{10} b^{10} \\
-(a b-(3 a b-4))=2 a b-4 \\
(a b)^{3}\left(a^{-1}+b^{-1}\right)=(a b)^{2}(a+b) \\
(a-b)^{3}=a^{3}-3 a^{2} b+3 a b^{2}-b^{3} \\
2 x^{2}-3 x-2=(2 x+1)(x-2) \\
\frac{1}{2} x+13=0 \Longrightarrow x=-26 \\
G=\{(x, y) \mid y=f(x)\} \\
f(x)=m x+b
\end{gathered}
$$

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3. Basic Algebra, Part I

Algebra is the language of mathematics, engineering and the sciences. We use algebra to express our thoughts, ideas, and to communicate with others who understand the language of algebra.

Algebra is a language in which we can precisely pose ourselves questions; Algebra is a set of tools for answering those questions.

## - You are manipulating Numbers

When we manipulate algebraic quantities, we are, in fact, manipulating numbers. This point must be ever kept in mind. The rules for manipulating algebraic quantities reflect the properties of the number system. This is an important point. Therefore, when you manipulate symbolic quantities in a questionable way, you must ask yourself the question, "Is what I have just done valid when I replace the symbols by numbers?"

This should be your guiding principle.

Section 3: Basic Algebra, Part I

A Fundamental Guiding Principle of Algebra:
"Is the algebraic manipulation that I just performed valid when I replace the symbols by numbers?"

### 3.1. The Basics

In this section we take a brief survey of the arithmetical operations and some of the properties of these operations that are exploited to perform algebraic manipulations.

## - The Arithmetical Operations

Let the letters $a, b$, and $c$ represent (real) numbers. As you well know we can add, subtract, multiply, and divide these numbers. In algebra, these operations are carried out symbolically:

1. Addition. The sum of $a$ and $b$ is $a+b$. The number $a$ is called a term of the expression $a+b$. (Of course, $b$ is a term too.)
2. Subtraction. The difference of $a$ and $b$ is $a-b$. The number $a$ is called a term of $a-b$. (Of course, $b$ is a term too.)
3. Mulitplication. Product of $a$ and $b$ is $a b$ or $a \cdot b$, or sometimes, $a \times b$. The numbers $a$ and $b$ are called factors of the product $a b$.
4. Division. The quotient of $a$ by $b$ is $\frac{a}{b}$, or $a / b$, or less frequently (in algebra), $a \div b$. Division is only defined when the denominator $b \neq 0$. The number $a$ is the numerator, and $b$ is the denominator.

Any combination of symbols and numbers related by the above arithmetical operations is called an algebraic expression.

## - Terms vs Factors

You should be particularly sensitive to the meaning of term and factor. Roughly, a term of an algebraic expression is a symbol that is connected to the rest of the expression by the operations of addition or subtraction. A factor of an algebraic expression is a symbol that is connected to the rest of the expression by the operations of multiplication.

Here are some examples of using the terminology described.

## Illustration 1.

(a) The number $x$ is a factor of the expression $2 x(x+1)$. Both 2 and $(x+1)$ are factors of the expression $2 x(x+1)$ as well.
(b) The symbol $w$ is a term of the expression $6 a+w$. The algebraic quantity $6 a$ is also a term of the expression.
(c) Consider the ratio, $\frac{x y}{a+b}$. The symbol $x$ is a factor of the numerator. The symbol $a$ the not a factor of the denominator, it is a term of the denominator.
(d) In the product $x^{3}(y+1)^{4}, x^{3}$ is a factor, as is $(y+1)^{4}$. Note that $y$ is not a factor of $x^{3}(y+1)^{4}$.

It is important that you be able to recognize factors and terms of algebraic expression. This is often key to simplifying an expression.

Quiz. Answer each of the questions 1-8. Passing is $100 \%$.

1. How many terms does the expression $4 x^{3}-x y+x y z^{2}$ ?
(a) 1
(b) 2
(c) 3
(d) 4
2. Is $x^{2}$ a factor of $x(x+1)$ ?
(a) Yes
(b) No
3. Is $x y$ a factor of the second term of $x+x y^{2}+x z^{2}$ ?
(a) Yes
(b) No
4. Is $x^{2}$ a factor of both the numerator and denominator of the expression $\frac{\left(1+y^{3}\right) x^{3}}{6 y^{3} x^{2}}$ ?
(a) Yes
(b) No
5. Is $x^{3}$ a factor of both the numerator and denominator of the expression $\frac{\left(1+y^{3}\right) x^{3}}{6 y^{3} x^{2}}$ ?
(a) Yes
(b) No
6. Is $y^{3}$ a factor of both the numerator and denominator of the expression $\frac{\left(1+y^{3}\right) x^{3}}{6 y^{3} x^{2}}$ ?
(a) Yes
(b) No
7. Is $x^{2}$ a common factor to all terms of the expression $x^{3} y-4 x^{6}+$ $3 x^{7 / 2} y^{3}$ ?
(a) Yes (b) No

Section 3: Basic Algebra, Part I
8. Is $x y$ a common factor to all terms of the expression $x y^{3}-$ $5 x^{3} y^{2}+1$.
(a) Yes (b) No

## EndQuiz

A Tip. Get into the habit of automatically identifying the factors and terms of an algebraic expression. Factors that are common to both numerator and denominator can be canceled; factors that are common to all terms of an expression can be factored out.

## - Parentheses, Brackets, and Braces

Throughout Lesson 1 and Lesson 2 I have utilized, on occasion, parentheses to group an expression that is to be treated as a unit. It is important for you to acquire the ability to correctly use the grouping mechanisms of (parentheses), [brackets], and \{braces\}.

When we want to square the quantity $x+y$ we would use the notation $(x+y)^{2}$. Without the parentheses the meaning of $x+y^{2}$ has an entirely different meaning.

Another example of same type is the squaring of the number -3 . That would be $(-3)^{2}$. The number -3 consists of two symbols: the minus sign ' - '; and the number ' 3 '. These two go together as a unit, a single entity. That being the case, we would write $(-3)^{2}$. A subset of students would express the same calculation as $-3^{2}$, but, to someone conversant in algebra, this has an entirely different meaning:

$$
(-3)^{2}=(-3)(-3)=9 \quad \text { whereas } \quad-3^{2}=-(3 \cdot 3)=-9,
$$

completely different evaluations! Some students say, "I wrote it and I know what I meant by $-3^{2}$." That's not the point is it? The point is to communicate with others in a way that they will understand what you mean without you being there to explain what you meant in any given situation.

Quiz. A student of mine who is in the habit of writing $-3^{2}$ instead of $(-3)^{2}$ has the problem of writing "minus the number three squared" in algebraic notation. How will he write this down on paper?
(a) $-3^{2}$
(b) n.o.t.

Section 3: Basic Algebra, Part I
Quiz. Evaluate the expression $x+x^{2}$ at $x=-1$. Which of the following is the correct evaluation?
(a) -2
(b) 0
(c) 2
(d) n.o.t.

Illustration 2. Here are some representative examples of grouping. When there are "nested" groups, we use brackets and/or braces.
(a) $x\left[1+(x+1)^{1 / 2}\right]^{2}$.
(b) $\left(2 x^{2}\right)^{3}$.
(c) $(x+1)\left\{1+\left[1+(x+1)^{2}\right]^{2}\right\}^{2}$.

Exercise 3.1. To test your ability to read algebraic expressions that contain groupings, in Illustration 2, put
(a) Put $x=3$ in $x\left[1+(x+1)^{1 / 2}\right]^{2}$ and evaluate.
(b) Put $x=-2$ in $\left(2 x^{2}\right)^{3}$ and evaluate.
(c) Put $x=0$ in $(x+1)\left\{1+\left[1+(x+1)^{2}\right]^{2}\right\}^{2}$ and evaluate.

Passing score is $100 \%$.

Sometimes we write mathematics within the body of text. When we do this, we must be careful writing the expressions. Do not write, for example, $1 / 2+x$ when you actually mean $1 /(2+x)$. Do not write $x+y / z$ when you mean $(x+y) / z$. Use parentheses, brackets, and braces whenever there is the possibility of confusion about the meaning of the expression.

Exercise 3.2. Calculate the values of $1 / 2+x$ and $1 /(2+x)$ for the case of $x=1$. Do they evaluate to the same numerical value? If not, what is your conclusion?

## - How to Negate Correctly

Let $a$ be an algebraic symbol. The symbol $-a$ is the negation of $a$, or additive inverse of $a$. The symbol $-a$ has the property that

$$
\begin{equation*}
a+(-a)=0 \tag{1}
\end{equation*}
$$

Negation of a Positive Number: The negation of 3 is -3 , the negation of $\frac{2}{3}$ is $-\frac{2}{3}$. These are well known observations.

Section 3: Basic Algebra, Part I
The Negation of a Negative Number: The negation of -6 is $-(-6)=6$. The negation of $-\sqrt{3}$ is $-(-\sqrt{3})=\sqrt{3}$. These follow from the well known fact that a "minus times a minus is a plus."

In general, we have the following principle:

$$
\begin{equation*}
-(-a)=a \tag{2}
\end{equation*}
$$

The negation of $a$ can also be thought of as $(-1) \cdot a$. That is,

$$
\begin{equation*}
-a=(-1) \cdot a \tag{3}
\end{equation*}
$$

It is from this interpretation of negation that equation (2) comes. Indeed,

$$
-(-a)=(-1)((-1) a)=(-1)(-1) a=a .
$$

This interpretation is important when we make calculations such as $(-a)^{5}$ or $(-a)^{2 k}$, where $k \in \mathbb{Z}$ in an integer. These kinds of evaluations were discussed in Lesson 2, in the section entitled The First Exponential Law; in particular, in that section, see the illustration for additional examples.

Section 3: Basic Algebra, Part I
Exercise 3.3. Using equation (3) and the Law of Exponents, simplify each of the following.
(a) $(-x)^{5}$
(b) $(-x)^{2 k}$
(c) $\left[\frac{-x^{3} y^{2}}{z^{4}}\right]^{3}$

Negating a grouped algebraic expression is always a problem for students. Here are some simple rules.

Negating a Grouped Expressions:

$$
\begin{align*}
& -(a+b)=-a-b \\
& -(a-b)=-a+b \tag{4}
\end{align*}
$$

In words, to negate the sum or difference of an expression, simply change the sign of each term.

Illustration 3. Examples of negating sum/differences of terms.
(a) $-(3 x+4 x y)=-3 x-4 x y$.
(b) $-\left(-4 x y+3 x^{2}\right)=4 x y-3 x^{2}$.

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(c) $-(a-b+c)=-a+b-c$.
(d) Mulitple Nesting. $-\left[-x-\left(4 x-2 x^{2}\right)\right]=-\left[-x-4 x+2 x^{2}\right]=$ $x+4 x-2 x^{2}=5 x-2 x^{2}$. When you have nested expressions, it is best to work your way from the inner-most level to the outer most level of nesting.

A Common Error. Students often make sign errors because of poor use of (parentheses), [brackets], and \{braces\}. If a student writes $-3 x-4 y$ when, in reality, the correct expression should have been $-(3 x-4 y)$. The first rendering, $-3 x-4 y$ does not simplify anymore whereas, $-(3 x-4 y)$ simplifies to $-3 x+4 y$.
The next example illustrates two approaches to simplifying an expression.

Example 3.1. Simplify $-[x-(-4 x+y)]$.
Exercise 3.4. Simplify each of the following expressions.
(a) $-(-3 x+5 y)$
(b) $-\left(9-(-2 x)^{3}\right)$
(c) $-(x y-(3 x y-4))$
(d) $\left[1-(-1)^{3}\right]-\left[\left(\frac{1}{2}\right)^{3}-\left(-\frac{1}{2}\right)^{3}\right]$

Section 3: Basic Algebra, Part I
Tip. Use (parentheses), [brackets], and \{braces\} liberally to "delimit" your expressions. This is especially important when dealing with the problems of negating and subtraction. Write $-(a+b)$ when you want to "negate" the expression $a+b$; write $x-(a+b)$ when you want to subtract $a+b$ from $x$.

- How to Invert Correctly

We have seen in Lesson 2 that

$$
\begin{equation*}
a^{-n}=\frac{1}{a^{n}} \quad n \in \mathbb{Z} \tag{5}
\end{equation*}
$$

In particular, when $n=1$ we obtain,

$$
\begin{equation*}
a^{-1}=\frac{1}{a} \tag{6}
\end{equation*}
$$

The expression $a^{-1}$ or $1 / a$ is called the (multiplicative) inverse of $a$. (Of course, we are assuming $a \neq 0$.) The number $a^{-1}$ has the property that

$$
a^{-1} \cdot a=1 \quad \text { or } \quad \frac{1}{a} \cdot a=1 .
$$

Section 3: Basic Algebra, Part I
In this (hopefully, short) section we discuss the basics of inverting algebraic expressions.

Problems are encountered we try to invert fractions. Here is the guiding equation:

$$
\begin{equation*}
\left(\frac{a}{b}\right)^{-1}=\frac{b}{a} \tag{7}
\end{equation*}
$$

Let's see how, operationally, this formula is used.
Illustration 4. Study the following examples.
(a) $\left(\frac{4}{5 x}\right)^{-1}=\frac{5 x}{4}$ by (7).
(b) $\left(\frac{x+1}{x+y}\right)^{-1}=\frac{x+y}{x+1}$, by (7).

Some important principles can be extracted from equation (7).

1. Inverse of an Inverse: For any $a \neq 0$,

$$
\left(a^{-1}\right)^{-1}=\left(\frac{1}{a}\right)^{-1}=a
$$

Thus,

$$
\left(a^{-1}\right)^{-1}=a
$$

This equation states that $a$ is the multiplicative inverse of $a^{-1}$ and really does not represent anything new. According to the Law of Exponents Law \#3, we would simplify $\left(a^{-1}\right)^{-1}$ normally by multiplying the exponents together to obtain $a$.
2. The inverse of $a$ fraction. Let $a \neq 0$ and $b \neq 0$, then

$$
\frac{1}{\frac{a}{b}}=\left(\frac{a}{b}\right)^{-1}=\frac{b}{a}
$$

Thus,

$$
\begin{equation*}
\frac{1}{\frac{a}{b}}=\frac{b}{a} . \tag{8}
\end{equation*}
$$

Section 3: Basic Algebra, Part I
This equation will be used to simplify ratios of fractions-stay tuned.

Here are some examples of the use of equation (8). Read the solutions carefully and try to understand the use of equation (8).

Example 3.2. Simplify each of the following:

$$
\text { (a) } \frac{1}{x^{3} / y^{2}} \text { (b) } \frac{1}{x /(x+y)} \text { (c) } \frac{1}{x /(x+y)^{-3}} \text {. }
$$

Illustration 5. Equation (5) can be used when negative powers other than -1 are involved.
(a) Consider the following general result.

$$
\begin{aligned}
\left(\frac{a}{b}\right)^{-n} & =\left[\left(\frac{a}{b}\right)^{-1}\right]^{n} & & \triangleleft \text { Law Exp. \#3 } \\
& =\left[\frac{b}{a}\right]^{n} & & \triangleleft \text { by }(7) \\
& =\frac{b^{n}}{a^{n}} & & \triangleleft \text { Law Exp. } \# 2
\end{aligned}
$$

Section 3: Basic Algebra, Part I
That is, we "flip the fraction" and raise each to the power $n$.

$$
\begin{equation*}
\left(\frac{a}{b}\right)^{-n}=\frac{b^{n}}{a^{n}} \tag{9}
\end{equation*}
$$

(b) Here is a simple example:

$$
\left(\frac{x+y}{z}\right)^{-3}=\frac{z^{3}}{(x+y)^{3}}
$$

(c) The numerator and denominator may themselves have powers in them. In that case we would also use Law $\# 3$ of the Law of Exponents. A particular manifestation of this formula is

$$
\left[\frac{(x+y)^{4}}{z^{3}}\right]^{-3}=\frac{z^{9}}{(x+y)^{12}}
$$

The above calculation can be easily executed by the inclusion of an intermediate step:

$$
\left[\frac{(x+y)^{4}}{z^{3}}\right]^{-3}=\left[\frac{z^{3}}{(x+y)^{4}}\right]^{3}=\frac{z^{9}}{(x+y)^{12}}
$$

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One can remove the negative exponent by inverting the fraction, then we can cube the result.
(d) Another example of the same type:

$$
\left[\frac{x^{5 / 4}}{y^{1 / 3}}\right]^{-4}=\left[\frac{y^{1 / 3}}{x^{5 / 4}}\right]^{4}=\frac{y^{4 / 3}}{x^{5}}
$$

Now it's your turn. Solve each of the following without error. My solutions may not be identical to yours. You may have to additional algebraic steps to verify your own answer. Try to justify each step.

Exercise 3.5. Simplify each of the following by eliminating all negative exponents using equations (6), equation (8) and the technique of "flipping the fraction" described in equation (9).
(a) $\left[\frac{x y+1}{x+y}\right]^{-1}$
(b) $\left[\frac{(x+y)^{4}}{x^{1 / 2} y^{3 / 2}}\right]^{-4}$
(c) $\left[\frac{a b^{-1}}{(a-b)^{-4}}\right]^{-2}$
(d) $\left(x^{3} y^{4}\right)^{-2}$
(e) $\frac{1}{x^{2} / y^{4}}$
(f) $\left[\frac{-x^{2}}{y^{3} / z^{4}}\right]^{3}$

### 3.2. How to Add/Subtract Algebraic Expressions

In this section we discuss methods of adding and subtracting algebraic expression that do not involve operation of division. When adding and/or subtracting the terms of an algebraic expression, we combine similar terms.

Adding Similar Terms:
To add similar terms, just add their numerical coefficients.

Illustration 6.
(a) $4 x^{3}+6 x^{3}=(4+6) x^{3}=10 x^{3}$. Here, we have just added the numerical coefficients.
(b) $6 x y^{3}-2 x y^{3}=(6-2) x y^{3}=4 x y^{3}$.
(c) $9\left(x^{2}+1\right)+6\left(x^{2}+1\right)=15\left(x^{2}+1\right)$.

Exercise 3.6. Combine each of the following.
(a) $5 x y z^{2}+6 x y z^{2}-8 x y z^{2}$ (b) $6 x \sqrt{x}-\frac{3}{2} x \sqrt{x}$

Section 3: Basic Algebra, Part I

Parentheses are used to group similar terms together in more complex expressions. Within each group, the problem of combining is at the same Skill Level as the previous illustration and exercise!

## Illustration 7.

(a) $5 x-6 y+12 x+\frac{1}{2} y=(5 x+12 x)+\left(-6 y+\frac{1}{2} y\right)=17 x-\frac{11}{2} y$.
(b) Combine $6 x y+7 x+3 \sqrt{x}+x y-\sqrt{x}$.

$$
\begin{aligned}
6 x y+7 x+3 \sqrt{x}+x y-\sqrt{x} & =(6 x y+x y)+7 x+(3 \sqrt{x}-\sqrt{x}) \\
& =7 x y+7 x+2 \sqrt{x} .
\end{aligned}
$$

Exercise 3.7. Combine each of the following expressions. As one of your step, rearrange and group similar terms using parentheses.
(a) $3 x^{2} y^{2}+5 x y+6 x y-x^{2} y^{2}$ (b) $-4 a b^{5}+6 a+6 a b^{5}-2 a$

This is not difficult in and of itself. It's a matter of identifying similar terms having, possibly different numerical coefficients, and adding those coefficients together. Another wrinkle in this process is the problem of subtraction of terms. This is taken up in the next paragraph.

Section 3: Basic Algebra, Part I

- Adding/Subtracting Grouped Expressions

There are a couple of basic rule for combining grouped expressions.

Add/Subtract Groups:

$$
\begin{align*}
& a+(b+c)=a+b+c \\
& a-(b+c)=a-b-c \tag{10}
\end{align*}
$$

Procedure. The basic procedure for adding or subtracting algebraic expressions is

1. remove the grouping parentheses using equations (10);
2. rearrange and regroup similar terms, if any;
3. combine similar terms by adding numerical coefficients.

Example 3.3. Combine each of the following.
(a) $\left(4 x^{3}+3 x y-2 x^{2}\right)+\left(5 x^{3}+9 x y-3 x^{2}\right)$
(b) $(8 a b-2 a-4 b)-(-4 a b+a-3 b)$

Section 3: Basic Algebra, Part I

Exercise 3.8. Combine each of the following expression. The principle tools to be used: The standard procedure for combining grouped expressions, equations (10) and equation (2).
(a) $(4 x-6 y)+(7 y-2 x)$
(b) $\left(5 x y^{2}-3 x^{2} y+2 x\right)-\left(3 x^{2} y-7 x y^{2}\right)$
(c) $(a-b+c)-(3 a-4 b+c)-(4 a+b)$.
(d) $4 x-[6 x-(12 x+3)]$.

Exercise 3.9. Part (d) of Exercise 3.8 was an example of nested parentheses. In case you missed that one, here are a couple more to practice on. Passing grade is $100 \%$.
(a) $4 a b-[5 a b+(4-3 a b)]$
(b) $x-(4 x+2 y)-[6 x-2 y-(4 x-4 y)]$
(c) $x-(4 x+2 y)-[-(6 x-2 y)-(4 x-4 y)]$

You have reached the end of Lesson 3. In the next lesson, we'll take up multiplication, expanding, and combining fractions by getting a

Section 3: Basic Algebra, Part I
common denominator. Many of the topics of this lesson will be revisited in the next. See you in Lesson 4!

## Solutions to Exercises

3.1. Solutins: Carefully remove each layer of parentheses.
(a) Putting $x=3$ in $x\left[1+(x+1)^{1 / 2}\right]^{2}$ becomes

$$
\begin{aligned}
3\left[1+(1+3)^{1 / 2}\right]^{2} & =3\left[1+4^{1 / 2}\right]^{2} \\
& =3[1+2]^{2}=3[3]^{2} \\
& =3(9)=27
\end{aligned}
$$

(b) Putting $x=-2$ in $\left(2 x^{2}\right)^{3}$ becomes

$$
\left(2(-2)^{2}\right)^{3}=(2(4))^{3}=8^{3}=512
$$

(c) Putting $x=0$ in $(x+1)\left\{1+\left[1+(x+1)^{2}\right]^{2}\right\}^{2}$ becomes

$$
\begin{aligned}
(0+1)\left\{1+\left[1+(0+1)^{2}\right]^{2}\right\}^{2} & =\left\{1+[1+1]^{2}\right\}^{2}=\left\{1+2^{2}\right\}^{2} \\
& =\{1+4\}^{2}=\{5\}^{2}=25
\end{aligned}
$$

Did you get them all? Were you slow and methodical?

Solutions to Exercises (continued)
3.2. Solution:

At $x=1,1 / 2+x$ becomes $1 / 2+1=3 / 2=\frac{3}{2}$.
At $x=1,1 /(2+x)$ becomes $1 /(2+1)=1 / 3=\frac{1}{3}$.
The values of these two expressions are different at $x=1$; therefore, I conclude that $1 / 2+x$ and $1 /(2+x)$ are not equivalent. I resolve not to equate them in the future!

Solutions to Exercises (continued)
3.3. Solutions:

$$
\begin{aligned}
& \text { (a) }(-x)^{5}=((-1) x)^{5}=(-1)^{5} x^{5}=-x^{5} . \\
& \text { (b) }(-x)^{2 k}=(-1)^{2 k} x^{2 k}=\left((-1)^{2}\right)^{k} x^{2 k}=1^{k} x^{2 k}=x^{2 k} . \\
& \text { (c) }\left[\frac{-x^{3} y^{2}}{z^{4}}\right]^{3}=\frac{(-1)^{3}\left(x^{3}\right)^{3}\left(y^{2}\right)^{3}}{\left(z^{4}\right)^{3}}=\frac{-x^{9} y^{6}}{z^{12}} .
\end{aligned}
$$

Exercise Notes: Problem (a) contains a few more details, problems (b) and (c) accelerates the process of simplifying: We go immediately from $x^{2 k}$, for example, to $(-1)^{2 k} x^{2 k}$.

Solutions to Exercises (continued)
3.4. Solutions:
(a) $-(-3 x+5 y)=3 x-5 y$.
(b) $-\left(9-(-2 x)^{3}\right)=-\left(9-\left(-8 x^{3}\right)\right)=-\left(9+8 x^{3}\right)=-9-8 x^{3}$.
(c) $-(x y-(3 x y-4))=-(x y-3 x y+4)=-(-2 x y+4)=2 x y-4$.
(d) Consider the following calculation:

$$
\begin{aligned}
{\left[1-(-1)^{3}\right]-\left[\left(\frac{1}{2}\right)^{3}-\left(-\frac{1}{2}\right)^{3}\right] } & =[1-(-1)]-\left[\frac{1}{8}-\left(-\frac{1}{8}\right)\right] \\
& =[1+1]-\left[\frac{1}{8}+\frac{1}{8}\right] \\
& =2-\frac{1}{4} \\
& =\frac{7}{4}
\end{aligned}
$$

If you carefully apply the rules you will surely not error. It's only when you become the master of algebra can you accelerate the simplification process. Take it slow and methodical.

Exercise 3.4.

Solutions to Exercises (continued)
3.5. Solutions:
(a) $\left[\frac{x y+1}{x+y}\right]^{-1}=\frac{x+y}{x y+1}$.
(b) $\left[\frac{(x+y)^{4}}{x^{1 / 2} y^{3 / 2}}\right]^{-4}=\left[\frac{x^{1 / 2} y^{3 / 2}}{(x+y)^{4}}\right]^{4}=\frac{x^{4 / 2} y^{12 / 2}}{(x+y)^{16}}=\frac{x^{2} y^{6}}{(x+y)^{16}}$.
(c) $\left[\frac{a b^{-1}}{(a-b)^{-4}}\right]^{-2}=\left[\frac{(a-b)^{-4}}{a b^{-1}}\right]^{2}=\frac{(a-b)^{-8}}{a^{2} b^{-2}}=\frac{b^{2}}{a^{2}(a-b)^{8}}$.
(d) $\left(x^{3} y^{4}\right)^{-2}=\frac{1}{\left(x^{3} y^{4}\right)^{2}}=\frac{1}{x^{6} y^{8}}$.
(e) $\frac{1}{x^{2} / y^{4}}=\frac{y^{4}}{x^{2}}$.
(f) $\left[\frac{-x^{2}}{y^{3} / z^{4}}\right]^{3}=\left[\frac{-x^{2} z^{4}}{y^{3}}\right]^{3}=\frac{\left(-x^{2}\right)^{3}\left(z^{4}\right)^{3}}{\left(y^{3}\right)^{3}}=\frac{-x^{6} z^{12}}{y^{9}}$.

Solutions to Exercises (continued)
3.6. Solutions: These are simple!
(a) $5 x y z^{2}+6 x y z^{2}-8 x y z^{2}=3 x y z^{2}$.
(b) $6 x \sqrt{x}-\frac{3}{2} x \sqrt{x}=\frac{9}{2} x \sqrt{x}$.

Solutions to Exercises (continued)
3.7. Solutions: 1 (a) Combine $3 x^{2} y^{2}+5 x y+6 x y-x^{2} y^{2}$ using parentheses to group similar terms:

$$
\begin{aligned}
3 x^{2} y^{2}+5 x y+6 x y-x^{2} y^{2} & =\left(3 x^{2} y^{2}-x^{2} y^{2}\right)+(5 x y+6 x y) \\
& =2 x^{2} y^{2}+11 x y
\end{aligned}
$$

1(b) Combine $-4 a b^{5}+6 a+6 a b^{5}-2 a$.

$$
\begin{aligned}
-4 a b^{5}+6 a+6 a b^{5}-2 a & =\left(-4 a b^{5}+6 a b^{5}\right)+(6 a-2 a) \\
& =2 a b^{5}+4 a
\end{aligned}
$$

Solutions to Exercises (continued)
3.8. Solutions:
(a) $(4 x-6 y)+(7 y-2 x)=(4 x-2 x)+(-6 y+7 y)=2 x+y$
(b) Combine $\left(5 x y^{2}-3 x^{2} y+2 x\right)-\left(3 x^{2} y-7 x y^{2}\right)$

$$
\begin{aligned}
\left(5 x y^{2}\right. & \left.-3 x^{2} y+2 x\right)-\left(3 x^{2} y-7 x y^{2}\right) \\
& =5 x y^{2}-3 x^{2} y+2 x-3 x^{2} y+7 x y^{2} \\
& =\left(5 x y^{2}+7 x y^{2}\right)+\left(-3 x^{2} y-3 x^{2} y\right)+2 x \\
& =12 x y^{2}-6 x^{2} y+2 x
\end{aligned}
$$

(c) Combine $(a-b+c)-(3 a-4 b+c)-(4 a+b)$.

$$
\begin{aligned}
& (a-b+c)-(3 a-4 b+c)-(4 a+b) \\
& \quad=a-b+c-3 a+4 b-c-4 a-b \\
& \quad=(a-3 a-4 a)+(-b+4 b-b)+(c-c) \\
& \quad=-6 a+2 b
\end{aligned}
$$

Solutions to Exercises (continued)
(d) Combine $4 x-[6 x-(12 x+3)]$.

$$
\begin{aligned}
4 x- & {[6 x-(12 x+3)] } \\
& =4 x-6 x+(12 x+3) \\
& =4 x-6 x+12 x+3 \\
& =(4 x-6 x+12 x)+3 \\
& =10 x+3
\end{aligned}
$$

Comments: The above solutions have more detail than is usually include when a master student of algebra performs. If you are not a master of algebra yet, take is slow and methodically. After a few problems, you can get the "feel" of the operations and you can accelerate you solutions. But don't do it until you are a master! Exercise 3.8.

## Solutions to Exercises (continued)

3.9. Solutions: In each case, you should have carefully removed parentheses.
(a) $4 a b-[5 a b+(4-3 a b)]$.

$$
\begin{aligned}
4 a b- & {[5 a b+(4-3 a b)] } \\
& =4 a b-5 a b-(4-3 a b) \\
& =4 a b-5 a b-4+3 a b \\
& =2 a b-4 \quad \triangleleft \text { accelerate a bit! }
\end{aligned}
$$

(b) $x-(4 x+2 y)-[6 x-2 y-(4 x-4 y)]$.

$$
\begin{aligned}
x- & (4 x+2 y)-[6 x-2 y-(4 x-4 y)] \\
& =x-4 x-2 y-6 x+2 y+(4 x-4 y) \\
& =x-4 x-2 y-6 x+2 y+4 x-4 y \\
& =(x-4 x-6 x+4 x)+(-2 y+2 y-4 y) \\
& =-5 x-4 y
\end{aligned}
$$

Solutions to Exercises (continued)

$$
\begin{aligned}
& \text { (c) } x-(4 x+2 y)-[-(6 x-2 y)-(4 x-4 y)] \\
& \qquad \begin{aligned}
x- & (4 x+2 y)-[-(6 x-2 y)-(4 x-4 y)] \\
& =x-4 x-2 y+(6 x-2 y)+(4 x-4 y) \\
& =x-4 x-2 y+6 x-2 y+4 x-4 y \\
& =(x-4 x+6 x+4 x)+(-2 y-2 y-4 y) \\
& =7 x-8 y
\end{aligned}
\end{aligned}
$$

Did you get them all right? It is important not to error! Concentrate! Concentrate! And ... oh yes, concentrate! Focus your mind on the problem at hand!

## Solutions to Examples

3.1. Two methods are demonstrated.

Method 1: Working from the outer most to the inner most.

$$
\begin{aligned}
-[x-(-4 x+y)] & =-x+(-4 x+y) \quad \triangleleft \text { Negate a sum } \\
& =-x-4 x+y \\
& =-5 x+y
\end{aligned}
$$

Method 2: Working from the inner most to the outer most.

$$
\begin{aligned}
-[x-(-4 x+y)] & =-[x+4 x-y] \quad \triangleleft \text { Negate a sum } \\
& =-[5 x-y] \\
& =-5 x+y
\end{aligned}
$$

Solutions to Examples (continued)
3.2. Solutions
(a) $\frac{1}{x^{3} / y^{2}}=\frac{y^{2}}{x^{3}}$.
(b) $\frac{1}{x /(x+y)}=\frac{x+y}{x}$. Notice that I have used parentheses to clearly define the expression under consideration.
(c) $\frac{1}{x /(x+y)^{-3}}=\frac{(x+y)^{-3}}{x}=\frac{1}{x(x+y)^{3}}$. In the last step, equation (5) was used.

Solutions to Examples (continued)
3.3. The principle tools are equations (10) equation (2). We follow the standard procedure:

Solution to (a):

$$
\begin{align*}
\left(4 x^{3}\right. & \left.+3 x y-2 x^{2}\right)+\left(5 x^{3}+9 x y-3 x^{2}\right) \\
& =4 x^{3}+3 x y-2 x^{2}+5 x^{3}+9 x y-3 x^{2}  \tag{S-1}\\
& =\left(4 x^{3}+5 x^{3}\right)+(3 x y+9 x y)+\left(-2 x^{2}-3 x^{2}\right)  \tag{S-2}\\
& =9 x^{3}+12 x y-5 x^{2} \tag{S-3}
\end{align*}
$$

Comments: Step (1) of the procedure was followed in (S-1); Step (2) was followed in (S-2), and (S-3) represents Step (3)

Solution to (b): The solution to this part follows the same pattern as part (a), so there will be fewer annotations.

$$
\begin{aligned}
& (8 a b-2 a-4 b)-(-4 a b+a-3 b) \\
& \quad=8 a b-2 a-4 b+4 a b-a+3 b \\
& \quad=(8 a b+4 a b)+(-2 a-a)+(-4 b+3 b) \\
& =12 a b-3 a-b
\end{aligned}
$$

Comments: In line (S-4), we removed the parentheses by following the standard rules. Here is more detail. The problem is basically of negation:

$$
\begin{aligned}
-(-4 a b+a-3 b) & =-(-4 a b)-a-(-3 b) & & \triangleleft \text { by }(10) \text { or }(4) \\
& =4 a b-a+3 b & & \triangleleft \text { from }(2)
\end{aligned}
$$

The last expression was then put back into equation (S-4) and the calculation continued.

## Important Points

Important Points (continued)
Right on! Let's have some discussion in case you missed the first time. Question:Is $x^{2}$ a common factor to all terms of the expression $x^{3} y-$ $4 x^{6}+3 x^{7 / 2} y^{3}$ ?

The answer is "Yes." Indeed, do a rewrite as follows.

$$
x^{3} y-4 x^{6}+3 x^{7 / 2} y^{3}=\underbrace{\left(x^{2}\right)(x y)}_{\text {first term }}-\underbrace{\left(x^{2}\right)\left(4 x^{4}\right)}_{\text {second term }}+\underbrace{\left(x^{2}\right)\left(3 x^{3 / 2} y^{3}\right)}_{\text {third term }}
$$

When written in this manner, you can now see that each term has the factor $x^{2}$ in common. It is a very important skill to have the ability to "spot" common factors of terms of an expression.

Yes, he is forced to write $-3^{2}$ to describe the operations of "minus the number three squared. But now he is in the sticky situation of using the same notation for two entirely different sequences of operations.

In the student's mind $1:-3^{2}$ means, in some situations, $(-3)(-3)=9$ and...

In the student's mind $2:-3^{2}$ means, in some situations, $-3 \cdot 3=-9$. Same notation, different results. Bad situation, indeed.

The correct answer is (b): The expression $x+x^{2}$ evaluates to 0 when we put $x=-1$; indeed,

$$
(-1)+(-1)^{2}=(-1)+1=0
$$

If your original answer was -2 you probably made the fundamental algebraic blunder just discussed. This calculation is wrong!

$$
\text { Wrong }!\Longrightarrow \quad-1-1^{2}=-1-1=2 \Longleftarrow \text { Wrong! }
$$

If you did this, DON'T DO IT AGAIN! $\mathfrak{W}$

